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N. V. Burmasheva, E. A. Larina, and E. Yu. Prosviryakov

Unidirectional Convective Flows of a Viscous Incompressible Fluid with Slippage in a Closed Layer

N. V. Burmasheva[1, 2, a)], E. A. Larina[2, b)], and E. Yu. Prosviryakov[1, 2, c)]

[1]*Institute of Engineering Science, Ural Branch of the Russian Academy of Sciences, 34 Komsomolskaya St., Ekaterinburg, 620049, Russian Federation*
[2]*B. N. Yeltsin Ural Federal University, 19 Mira St., Ekaterinburg, 620002, Russian Federation*

[a)]Corresponding author: nat_burm@mail.ru
[b)]larinakaterina@hotmail.com
[c)]evgen_pros@mail.ru

Abstract. An exact solution describing the flow of a viscous incompressible fluid in a closed horizontal layer is obtained in view of the Navier slip condition and the inhomogeneous pressure distribution on one of the layer boundaries. The principal possibility of velocity field stratification and the dependence of the number of stratification points on the slip length are demonstrated.

INTRODUCTION

One of the most frequently used models for describing convection in viscous heat-conducting fluids is the system of heat convection equations in the Boussinesq approximation [1-5]. As is the case with any other system of partial differential equations, its solution depends strongly on the boundary conditions. In this paper, with the aid of families of generalized solutions [6-10], a new exact solution is constructed for the unidirectional flow of a viscous incompressible fluid in a horizontal layer [10-21], with the Navier slip condition [22] satisfied on its rigid boundary. A distinctive feature of the constructed solution is taking into account zero fluid flow; i.e., in fact, it is a flow in a closed layer [23], with a non-uniform pressure distribution on the free surface. It is shown that, even if the boundaries are thermally insulated, the velocity field admits stratification regardless of the magnitude of the longitudinal pressure gradient and the slip length.

BOUNDARY VALUE PROBLEM FORMULATION

The steady-state unidirectional shear flow of a viscous incompressible fluid in a horizontal closed layer is considered. Such fluid motions are modeled using a system of heat convection equations. Given the Boussinesq hypothesis about the temperature dependence of fluid the density, this system of equations takes the form [1-5]

$$V_x \frac{\partial V_x}{\partial x} = -\frac{\partial P}{\partial x} + \nu\left(\frac{\partial^2 V_x}{\partial x^2} + \frac{\partial^2 V_x}{\partial z^2}\right);\ \frac{\partial P}{\partial z} = g\beta T;\ V_x \frac{\partial T}{\partial x} = \chi\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right);\ \frac{\partial V_x}{\partial x} = 0. \quad (1)$$

Here, $V_x = V_x(x,z)$ is the x-component of the velocity vector $\boldsymbol{V}$; $P(x,z)$ is deviation of pressure from hydrostatic, divided by average density; $T(x,z)$ is deviation of temperature from the reference value; ν, χ are the kinematic (molecular) viscosity of the fluid and the coefficient of thermal diffusivity; β is the temperature coefficient of volumetric expansion of the fluid; g is acceleration of gravity.

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The number of unknown functions in system (1) is smaller than the number of equations of this system. However, if we integrate the last equation (the incompressibility equation), we obtain that the flow velocity along the horizontal axis *Ox* must be described by a homogeneous function of the transverse coordinate *z* [10-12],

$$V_x = U(z). \tag{2}$$

As a result of the substitution of class (2) into system (1), we obtain a closed system of equations for the velocity *U*, temperature *T*, and pressure *P*,

$$\frac{\partial P}{\partial x} = \nu \frac{\partial^2 U}{\partial z^2}; \ \frac{\partial P}{\partial z} = g\beta T; \ U\frac{\partial T}{\partial x} = \chi\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial z^2}\right). \tag{3}$$

It can be strictly shown that the form of the equations of system (3) completely determines the structure of the solution class for the temperature field and the pressure field,

$$P = P_0(z) + xP_1(z); \ T = T_0(z) + xT_1(z). \tag{4}$$

Using the method of indefinite coefficients, by substituting class (2), (4) into system (3), we can reduce this system to the equivalent record

$$T_1'' = 0; \ P_1' = g\beta T_1; \ \nu U'' = P_1; \ \chi T_0'' = UT_1; \ P_0' = g\beta T_0. \tag{5}$$

System (5) is a system of ordinary differential equations; each function involved in these equations depends only on the transverse coordinate *z*.

We will further consider the following system of boundary conditions. The temperature at the lower boundary $z = 0$ is equal to the zero reference value. Besides, the Navier slip condition is specified on the lower boundary. In view of the structure of the chosen generalized class of solutions (2), (4), these conditions are written as

$$T_0\big|_{z=0} = 0; \ \alpha \frac{\partial U}{\partial z}\bigg|_{z=0} = U(0). \tag{6}$$

Here, α is the length of the slip.

On the upper (free) surface, $z = h$, the values of temperature and pressure are specified, as well as the horizontal temperature and pressure gradients. Also, at the upper boundary of the fluid layer, zero shear stress is set. Taking into account the structure of the class of solutions (2), (4), these conditions take the form

$$T_0\big|_{z=h} = \vartheta; \ T_1\big|_{z=h} = A; \ P_0(h) = S_0; \ P_1(h) = S_1; \frac{dU}{dz}\bigg|_{z=h} = 0. \tag{7}$$

In addition, we assume that the fluid flow is zero,

$$\int_0^h U dz = 0. \tag{8}$$

EQUATION SYSTEM SOLUTION

The exact solution to the boundary value problem (5)–(8) is polynomial,

$$T_1 = \frac{10S_1}{gh^2(h+5\alpha)}(h-z)(h+3\alpha) + \frac{A\beta}{4h(h+5\alpha)}\left(-11h^2 + 5h(3z-8\alpha) + 60z\alpha\right);$$

$$P_1 = -\frac{S_1}{h^2(h+5\alpha)}\left(4h^3 + 5hz(z-6\alpha) + 15z^2\alpha + 10h^2(-z+\alpha)\right) +$$

$$+\frac{Ag\beta}{8h(h+5\alpha)}(h-z)\left(h(7h-15z) + 20\alpha(h-3z)\right);$$

$$U = \frac{S_1}{12h^2\nu(h+5\alpha)}\left(-24h^3z^2 - 5hz^3(z-12\alpha) + 20h^2z^2(z-3\alpha) - 15z^4\alpha + 8h^4(z+\alpha)\right) +$$

$$+\frac{Ag\beta}{96h\nu(h+5\alpha)}\left(42h^3z^2 + 5hz^3(3z-32\alpha) + 60z^4\alpha - 12h^4(z+\alpha) + 4h^2z^2(-11z+30\alpha)\right). \quad (9)$$

The expressions for the background pressure P_0 and the background temperature T_0 are omitted here since they are too cumbersome. If necessary, they can be fairly easily obtained by integrating the last two equations of system (5) with due regard for the boundary conditions and the given exact solution (9).

The structure of the exact solution (9) shows that the flow velocity and the longitudinal pressure and temperature gradients, as well as the background pressure and the background temperature, are determined (besides the physical characteristics of the fluid) by the values of four parameters: gradient A, gradient S_1, slip length α and layer thickness h. Note also the nonlinear dependence of the exact solution (9) with respect to the parameter α.

INVESTIGATION OF THE SOLUTION

A distinctive feature of the exact solution (9) is taking into account the inhomogeneity of pressure distribution at the upper boundary of the fluid layer under study. This leads to the fact that, e.g., the flow velocity U, even in the absence of heat sources at the boundary (both boundaries are thermally insulated), can have stagnant points. Indeed, when the longitudinal temperature gradient A is zero, the velocity U is described by the expression

$$U = \frac{S_1}{12h^2\nu(h+5\alpha)}\left(-24h^3z^2 - 5hz^3(z-12\alpha) + 20h^2z^2(z-3\alpha) - 15z^4\alpha + 8h^4(z+\alpha)\right) =$$

$$= \frac{S_1}{12h^2\nu(h+5\alpha)}\left[\left(-24h^3z^2 - 5hz^4 + 20h^2z^3 + 8h^4z\right) + \alpha\left(60hz^3 - 60h^2z^2 - 15z^4 + 8h^4\right)\right] =$$

$$= \frac{S_1h^5}{12h^2\nu(h+5\alpha)}\left[\left(-24Z^2 - 5Z^4 + 20Z^3 + 8Z\right) + \frac{\alpha}{h}\left(60Z^3 - 60Z^2 - 15Z^4 + 8\right)\right]. \quad (10)$$

Here, $Z = z/h\varepsilon[0, 1]$ is the dimensionless transverse coordinate. It follows from Eq. (10) that the presence or absence of zero points for the velocity U is determined only by the value of the ratio α/h. The analysis of the behavior of the polynomials in expression (10) allows us to state that, for any value of the parameter α/h, the velocity U can have one zero point regardless of the pressure gradient value ($S_1 \neq 0$). Obviously, by virtue of Eq. (10), the velocity U will be identically equal to zero when $S_1 \equiv 0$.

We rewrite the velocity expression (9) for the case that the temperature gradient A is different from zero in the following form:

$$U = \frac{Ag\beta}{96h\nu(h+5\alpha)}\left\{k\left(-24Z^2 - 5Z^3\left(Z - 12\frac{\alpha}{h}\right) + 20Z^2\left(Z - 3\frac{\alpha}{h}\right) - 15Z^4\frac{\alpha}{h} + 8\left(Z + \frac{\alpha}{h}\right)\right) +\right.$$

$$\left. + \left(42Z^2 + 5Z^3\left(3Z - 32\frac{\alpha}{h}\right) + 60Z^4\frac{\alpha}{h} - 12\left(Z + \frac{\alpha}{h}\right) + 4Z^2\left(-11Z + 30\frac{\alpha}{h}\right)\right)\right\}.$$

Here, k is defined as

$$k = \frac{8S_1}{Ag\beta h};$$

this ratio characterizes the contribution of the Poiseuille flow, as compared with the thermogravitational one, to the formation of the resulting flow velocity U. Figure 1 depicts the locus of points satisfying the condition $U = 0$. It clearly illustrates that, at certain values of the coefficient k, the velocity U can have two zero points in the layer. The U velocity profiles for various values of the coefficient k are shown in Fig. 2.

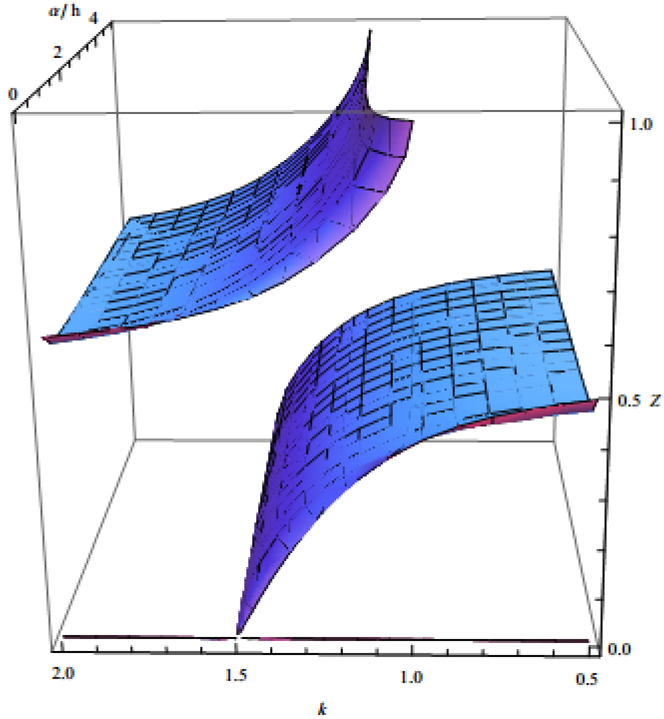

FIGURE 1. The set of points satisfying the condition $U = 0$

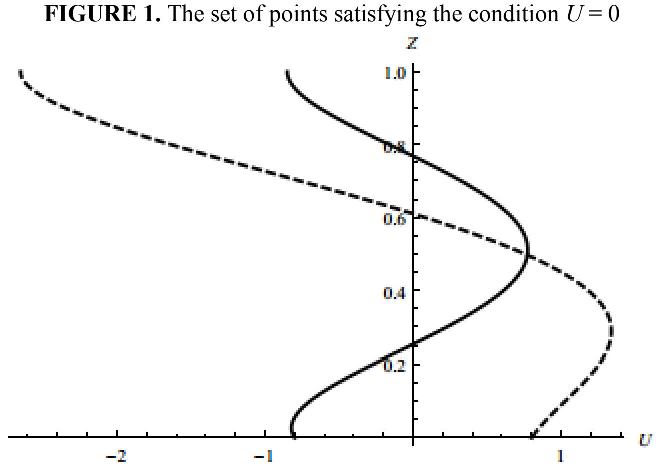

FIGURE 2. U velocity profiles ($\alpha/h = 0.5$) with $k = 1.7$ (dashed line) and $k = 1.3$ (solid line)

CONCLUSION

A new exact solution has been constructed for a closed horizontal layer, with the Navier slip condition satisfied on one of its boundaries. It has been demonstrated that, even in the case of thermally insulated boundaries, this solution allows the possibility of the appearance of stratification points in hydrodynamic fields due to the account taken of the nonzero longitudinal pressure gradient at the upper boundary of the layer.

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